Lecture 1
Automorphisms and derivations of $\mathbb{C}[x, y]$

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Abstract

Today we describe the group of automorphisms of $\mathbb{C}[x, y]$, an algebra of polynomials with two variables over the field of complex numbers. First description of this group was obtained by Heinrich Jung in 1942. He showed that any automorphism of $\mathbb{C}[x, y]$ is a composition of linear automorphisms $x \to a_1x + b_1y,\ y \to a_2x + b_2y$ and triangular automorphisms $x \to x,\ y \to y + p(x)$.

We will also describe all locally nilpotent derivations of $\mathbb{C}[x, y]$. This was done by Rudolf Rentschler in 1968.

Partial derivatives

Recall that there are two partial derivatives $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$ on the algebra $\mathbb{C}[x, y]$. If α is an automorphism of $\mathbb{C}[x, y]$ and $u = \alpha(x),\ v = \alpha(y)$ we can define partial derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ relative to u and v. Just write an element of $\mathbb{C}[x, y]$ as a polynomial in u and v and take the corresponding partial derivative. Now we have a lot of partial derivatives and they all have the following property.

If we apply any partial derivative to a polynomial sufficiently many times we obtain zero.

Indeed, any $f \in \mathbb{C}[x, y]$ can be recorded as a polynomial in u and v. Suppose that $\deg_u(f) = k$. Then $\frac{\partial^{k+1}}{\partial u^{k+1}}(f) = 0$ because when we apply $\frac{\partial}{\partial u}$ to f degree relative to u becomes smaller: $\deg_u(\frac{\partial f}{\partial u}) = k - 1$.

Partial derivatives and Jacobian

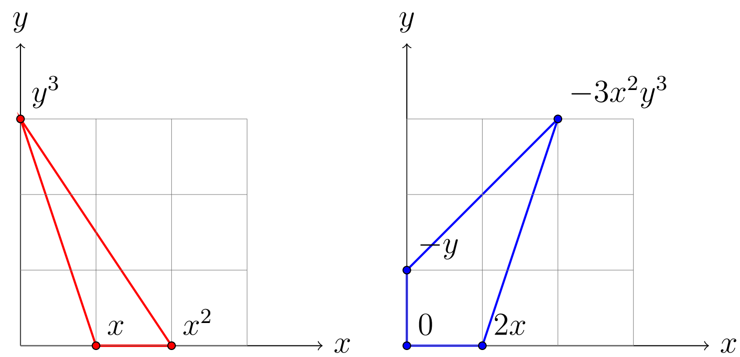
Recall what is the Jacobian of two element a and b of $\mathbb{C}[x, y]$:
$\mathrm{J}(a, b) = a_x b_y - a_y b_x$.
So $\mathrm{J}(x, b) = b_y = \partial_y(b)$ and $\mathrm{J}(b, y) = b_x = \partial_x(b)$.

Let us take now $\mathrm{J}(u, b)$. Of course it is $u_x b_y - u_y b_x$. On the other hand by the chain rule $\mathrm{J}_{x,y}(a, b) = \mathrm{J}_{x,y}(u, v)\mathrm{J}_{u,v}(a, b)$ and so $\mathrm{J}(u, b) = \mathrm{J}_{x,y}(u, v)b_v$.

Since $1 = \mathrm{J}_{x,y}(x, y) = \mathrm{J}_{x,y}(u, v)\mathrm{J}_{u,v}(x, y)$ and both $\mathrm{J}_{x,y}(u, v)$ and $\mathrm{J}_{u,v}(x, y)$ are elements of $\mathbb{C}[x, y]$ we see that $\mathrm{J}_{x,y}(u, v)$ is a non-zero complex number. We can assume that it is 1: just divide, say u by an appropriate number. Then $b_v = \mathrm{J}(u, b)$.

Newton polygons and leading forms.

Newton suggested to represent a polynomial f in two variables by a picture on a plane which is called the Newton polygon of f. For example $\mathcal{N}(x+x^2-y^3)$ and $\mathcal{N}(1-y+2x-3x^2y^3)$ are



So to get the Newton polygon for $a \in \mathbb{C}[x,y]$ mark all points on the integer lattice of the plane which correspond to monomials of a.

Newton polygon $\mathcal{N}(a)$ of a has edges. Take any edge e of $\mathcal{N}(a)$ and consider only those monomials of a which correspond to the points of e. Then we can write $a = f + a_1$ where $\mathcal{N}(f) = e$ and $\mathcal{N}(a_1) \bigcap e = \emptyset$. Polynomial f is called the leading form of a (corresponding to the edge e). The leading forms can also be defined for the vertices of $\mathcal{N}(a)$ in the same way. The leading form of a can be defined for any direction. Actually, two leading forms correspond to a direction: there are two parallel lines in a given direction which "touch" a Newton polygon.

Let us look at our examples.
$\mathcal{N}(x+x^2-y^3)$ is a triangle and $\mathcal{N}(1-y+2x-3x^2y^3)$ is quadrilateral.
So we have six different leading forms for $x+x^2-y^3$: $x,\ x^2,\ -y^3;\ x+x^2,\ x^2-y^3$, and $x-y^3$ and eight leading forms for $1-y+2x-3x^2y^3$

Of course the leading forms for different directions can correspond to the same vertex of $\mathcal{N}(a)$.

We will be interested only in forms corresponding to vertices and edges facing infinity of the first quadrant.

Later we will introduce weight degree functions and non-geometric definition of the leading forms.

Lnds and leading forms.

In analysis we talk about derivatives, in algebra we talk about derivations. Say, if we define $\partial_f(g)$ by $\partial_f(g) = \mathrm{J}(f,g)$ where f is not an image of x or y under an automorphism we cannot call ∂ a partial derivative, but we can call it derivation. Here is a definition: a function ∂ on any algebra A over a field K is called a derivation if it is linear, i.e. $\partial(k_1a_1 + k_2a_2) = k_1\partial(a_1) + k_2\partial(a_2)$ and satisfies the Leibniz rule: $\partial(a_1a_2) = \partial(a_1)a_2 + a_1\partial(a_2)$.

Indeed,
$\mathrm{J}(f, c_1a_1 + c_2a_2) = f_x(c_1a_1 + c_2a_2)_y - f_y(c_1a_1 + c_2a_2)_x = c_1\mathrm{J}(f,a_1) + c_2\mathrm{J}(f,a_2)$ and
$\mathrm{J}(f, a_1a_2) = f_x(a_1a_2)_y - f_y(a_1a_2)_x = f_x(a_{1y}a_2 + a_1a_{2y}) - f_y(a_{1x}a_2 + a_1a_{2x}) = \mathrm{J}(f,a_1)a_2 + a_1\mathrm{J}(f,a_2)$.

Let us fix any direction and denote the corresponding leading form of $a \in \mathbb{C}[x,y]$ by $\overline{a}$.

Since $(x^iy^j) \times (x^ky^l) = x^{i+k}y^{j+l}$ it is clear that $\overline{ab} = \overline{a}\overline{b}$ for any a, $b \in \mathbb{C}[x,y]$.

Since $\mathrm{J}(x^iy^j, x^ky^l) = (il - jk)x^{i+k}y^{j+l}x^{-1}y^{-1}$ it is also true that $\overline{\mathrm{J}(a,b)} = \mathrm{J}(\overline{a}, \overline{b})$ if $\mathrm{J}(\overline{a}, \overline{b}) \neq 0$: product of monomials is never zero, but Jacobian of monomials is zero on occasions.

Recall that if $u = \alpha(x)$ where α is an automorphism of $\mathbb{C}[x,y]$ then $\partial_u(b) = \mathrm{J}(u,b)$ is a derivation with an additional property: if we apply this derivation to an element $b \in \mathbb{C}[x,y]$ sufficiently many times the result will be zero.

Any derivation on A with this property is called a locally nilpotent derivation, lnd for short. So any partial derivative of $\mathbb{C}[x,y]$ is an lnd.

A derivation $\overline{\partial} = \partial_{\overline{u}}$ is also an lnd ($\partial_{\overline{u}}(b) = \mathrm{J}(\bar{u}, b)$). To see this let us present $b \in \mathbb{C}[x,y]$ as a sum of several forms. First, $b = \bar{b} + b_1$. Then $b = \bar{b} + \overline{b_1} + b_2$, then $b = \bar{b} + \overline{b_1} + \overline{b_2} + b_3$, and so on. Let us call elements $\overline{b_i}$ homogeneous. For example, $\bar{u}$ is homogeneous.

Since $\bar{\partial}$ is linear it is sufficient to check that $\bar{\partial}$ is locally nilpotent on homogeneous elements.

As we saw above, the Jacobian of two homogeneous elements is also homogeneous (zero is a homogeneous element). Because of that if b is a homogeneous element and $\overline{\partial}^n(b)$ is not zero then $\overline{\partial}^n(b) = \overline{\partial^n(b)}$.

Indeed if b is homogeneous and $\mathrm{J}(\bar{u}, b) \neq 0$ then $\mathrm{J}(\bar{u}, b) = \overline{\mathrm{J}(u,b)}$. Since ∂_u is an lnd, $\partial_u^m(b) = 0$ for some m. Therefore $\overline{\partial}^n(b) = 0$ for some $n \leq m$.

Lnds and degrees.

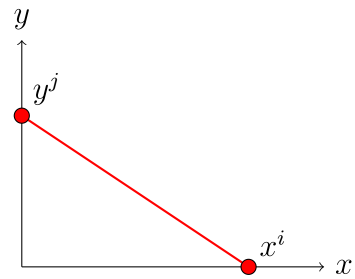
For an $a \in \mathbb{C}[x,y]$ its degree relative to x can be defined as the maximal number of times ∂_x can be applied before zero is obtained. Let ∂ be an lnd. We can define for any element $a \in \mathbb{C}[x,y]$ a function $\deg(a) = \max(n|\partial^n(a) \neq 0)$. It follows from the Leibniz rule that $\partial^n(ab) = \sum \binom{n}{i}\partial^i(a)\partial^{n-i}(b)$ which, of course, implies that $\deg(ab) = \deg(a) + \deg(b)$. It is very easy to check that deg has all familiar properties of the ordinary degree of polynomials.

Leading forms of u**.**

When Newton introduced Newton polygons he assumed that the origin is always a vertex of the polygon, i.e. that the free term of the polynomial is not zero. In this section we will make the same assumption to have "nicer" polygons.

Let us choose a direction so that $\overline{u}$ is a monomial cx^iy^j. Of course, $c \neq 0$. Then $\overline{\partial}(x^iy^j) = \mathrm{J}(cx^iy^j, x^iy^j) = 0$ for the corresponding $\overline{\partial}$. So $\deg(x^iy^j) = i\deg(x) + j\deg(y) = 0$ and $\deg(x) = \deg(y) = 0$ if both i and j are not zero. Hence both $\overline{\partial}(x) = -cjx^iy^{j-1} = 0$ and $\overline{\partial}(y) = cix^{i-1}y^j = 0$, i.e. $i = j = 0$ and $\bar{u} = c$. But then $u = c$ which is absurd: the image of x under an automorphism cannot be an element of $\mathbb{C}$.

Thus either i or j is zero and the Newton polygon of u is a right triangle with the right angle in the origin.



Of course, we can get a degenerate triangle which belongs to the x axis or y axis. Then $u = p(x)$ or $u = q(y)$ and, since u is the image of x under an automorphism, $p(x) = \alpha x + \beta$, $\alpha \neq 0$; $q(y) = \gamma y + \delta$, $\gamma \neq 0$. In the third case $\bar{u} = \lambda x^m + \cdots + \mu y^n$ and the corresponding edge of $\mathcal{N}(u)$ is parallel to the vector $\langle -m, n\rangle$. We can rewrite $\lambda x^m + \cdots + \mu y^n$ as $x^m p(z)$ where $p(z)$ is a polynomial in one variable $z = x^{-m_1}y^{n_1}$ where $\langle -m_1, n_1\rangle$ is "the shortest" vector with integral components going in the same direction as $\langle -m, n\rangle$.

Since $p(z) = \mu\prod(z - c_i)$ by the fundamental theorem of algebra we can write $\overline{u} = \mu x^m \prod(z - c_i) = \mu\prod(y^{n_1} - c_i x^{m_1})$.

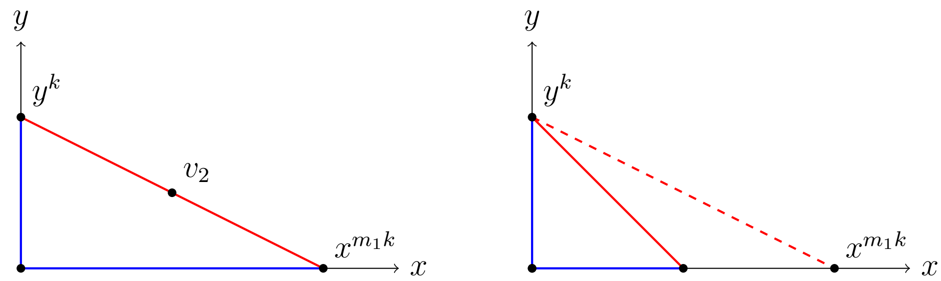
Let $\overline{\partial}$ be the corresponding lnd. Of course $\overline{\partial}(\overline{u}) = 0$ and if we choose the corresponding deg then $\deg(\overline{u}) = 0$. It implies that any factor of $\overline{u}$ also has degree zero. So $\deg(y^{n_1} - c_i x^{m_1}) = 0$. If we have two different c_i's in the factorization we will also get $\deg(x^{m_1}) = 0$ and $\deg(y^{n_1}) = 0$ and $\deg(x) = \deg(y) = 0$. But then $\overline{u}_x = \overline{u}_y = 0$ and $\overline{u}$ is a complex number which is impossible. So $\overline{u} = \mu(y^{n_1} - cx^{m_1})^k$ and $\overline{\partial}(y^{n_1} - cx^{m_1}) = 0$.

The end of the story.

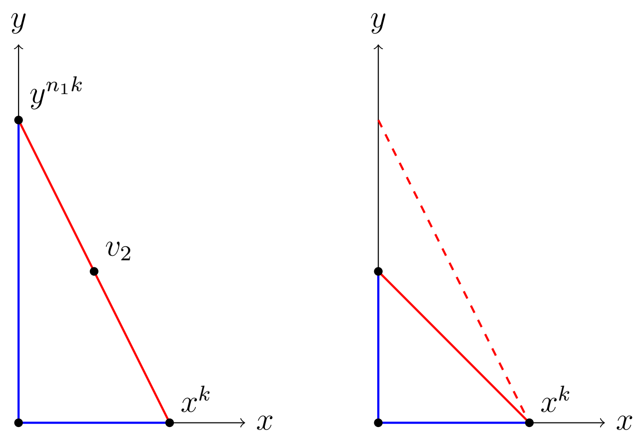
Since $\overline{\partial}(y^{n_1} - cx^{m_1}) = 0$ we have $n_1 y^{n_1-1}\overline{\partial}(y) = cm_1 x^{m_1-1}\overline{\partial}(x)$. Hence y^{n_1-1} divides $\overline{\partial}(x)$ and x^{m_1-1} divides $\overline{\partial}(y)$.

Put $d_x = \deg(x)$ and $d_y = \deg(y)$ where deg is the degree determined by $\bar{\partial}$. Then $(n_1 - 1)d_y \leq d_x - 1$ and $(m_1 - 1)d_x \leq d_y - 1$. Therefore $(m_1 - 2)d_x + (n_1 - 2)d_y \leq -2$ which is possible only if either m_1 or n_1 is 1.

If $n_1 = 1$ let us make an automorphism β_1 of $\mathbb{C}[x, y]$ which is given by $x \to x$ and $y \to y + cx^{m_1}$. Then $(y - cx^{m_1})^k \mapsto (y + cx^{m_1} - cx^{m_1})^k = y^k$. Therefore β_1 "collapses" the hypotenuse of $\mathcal{N}(u)$ to the left vertex $(0, k)$ corresponding to y^k. Here are the Newton polygons of u and $\beta_1(u)$:



If $m_1 = 1$ let us make an automorphism β_2 of $\mathbb{C}[x, y]$ which is given by $y \to y$ and $x \to x + c^{-1}y^{n_1}$. Then the hypotenuse of $\mathcal{N}(u)$ will collapse to the right vertex $(k, 0)$ corresponding to x^k. Here are the Newton polygons of u and $\beta_2(u)$:



In both case the Newton polygon became “smaller” than $\mathcal{N}(u)$. Using induction on, say, area of $\mathcal{N}(u)$ we conclude that after several “triangular” automorphisms like β_1 and β_2 the Newton polygon of the image of u will be either $\lambda_1 x + \nu_1$ or $\lambda_2 y + \nu_2$ where $\lambda_i,\ \nu_i \in \mathbb{C}$.

Denote the composition of these automorphisms by γ.

If $\gamma(u) = \lambda_1 x + \nu_1$ then $\gamma(v) = \mu_1 y + p(x)$ since $\mathrm{J}(\gamma(u), \gamma(v)) = \lambda\gamma(v)_y$ is a complex number.

Similarly, if $\gamma(u) = \lambda_2 y + \nu_2$ then $\gamma(v) = \mu_2 x + q(y)$ since $\mathrm{J}(\gamma(u), \gamma(v)) = -\lambda_2\gamma(v)_x$ is a complex number.

If we make one more triangular automorphism β for which $x \to x$ and $y \to y - \mu_1^{-1}p(x)$ then $\beta\gamma\alpha(x) = \lambda_1 x + \nu_1$ and $\beta\gamma\alpha(y) = \mu_1 y$. We also can get rid of ν_1 by a triangular automorphism. Of course, the other case can be treated exactly in the same manner.

Since the inverse of a triangular automorphism is also a triangular automorphism, we showed that any automorphism is a composition of triangular automorphisms and either an automorphism $x \to \lambda x$, $y \to \mu y$ or an automorphism $x \to \lambda y$, $y \to \mu x$.

This description is equivalent to the Jung's description.

H. W. E. Jung, Uber ganze birationale Transformationen der Ebene , J. Reine Angew. Math. 184 (1942), 161-174.

The idea to use lnd for a description of automorphisms belongs to Jacques Dixmier. See Dixmier, Jacques Sur les algèbres de Weyl. (French) Bull. Soc. Math. France 96, (1968), pages 209-242, where the group of automorphisms of the first Weyl algebra is described.

It was used by Rudolf Rentschler: Rentschler, Rudolf Opèrations du groupe additif sur le plan affine. (French) C. R. Acad. Sci. Paris Sèr. A-B 267, (1968), pages A384-A387, to describe all generalized shifts of the plane and the group of automorphisms of $\mathbb{C}[x, y]$.

Now we develop a bit of theory.

Definitions, notations and technical lemmas.

Here are some necessary notions and facts.

Let A be a $\mathbb{C}$-algebra. A $\mathbb{C}$-homomorphism ∂ of A is called a *derivation* of A if it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + a\partial(b)$.

A derivation is *irreducible* if $\partial(A)$ does not belong to a proper principal ideal. (So according to this definition zero derivation is irreducible!)

Any derivation ∂ determines two subalgebras of A. One is the kernel of ∂ which is usually denoted by A^∂ and is called the *ring of ∂-constants*.

The other is $\mathrm{Nil}_A(\partial)$, the *ring of nilpotency of* ∂. It is determined by $\mathrm{Nil}_A(\partial) = \{a \in A | \partial^n(a) = 0, n >> 1\}$. In other words $a \in \mathrm{Nil}_A(\partial)$ if for a sufficiently large natural number n we have $\partial^n(a) = 0$.

Both A^∂ and $\mathrm{Nil}_A(\partial)$ are subalgebras of A because of the Leibniz rule.

We will call a derivation *locally nilpotent* if $\mathrm{Nil}_A(\partial) = A$.

The best examples of locally nilpotent derivations are the partial derivatives on the rings of polynomials $\mathbb{C}[x_1, ..., x_n]$.

With the help of a locally nilpotent derivation acting on A, we can define a function $\deg_\partial$ by $\deg_\partial(a) = \max(n|\partial^n(a) \neq 0)$ if $a \in A^* = A \setminus 0$ and $\deg_\partial(0) = -\infty$.

Then the function $\deg_\partial$ is a degree function, i.e.,
$\deg_\partial(a + b) \leq \max(\deg_\partial(a), \deg_\partial(b))$ and
$\deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b)$.

Two locally nilpotent derivations are *equivalent* if the corresponding degree functions are the same.

By definition $\deg_\partial$ has only non-negative values on A^* and $a \in A^\partial \setminus 0$ if $\deg_\partial(a) = 0$. So it is clear that the ring A^∂ is *factorially closed*; i. e., if $a, b \in A^*$ and $ab \in A^\partial$, then $a, b \in A^\partial$.
(In the literature subrings with this property are called saturated multiplicatively closed subrings.)

Let F be the field of fractions of A. Any derivation ∂ can be extended to a derivation on F by the "calculus" formula $\partial(ab^{-1}) = (\partial(a)b - a\partial(b))b^{-2}$. We will denote this extended derivation also by ∂.

Lemma 1. Let ∂ be a locally nilpotent nonzero derivation of A. Then there exists an element $t \in F$ for which $\partial(t) = 1$ and $\mathrm{Nil}_F(\partial) = F^\partial[t]$.

Proof. ∂ is a nonzero derivation so $A \neq A^\partial$ and there exists an $a \in A \setminus A^\partial$. Put $r = \partial^n(a)$ where $n = \deg_\partial(a) - 1$. Then $r \notin A^\partial$ and $\partial(r) \in A^\partial$. If $t = \frac{r}{\partial(r)}$ then $\partial(t) = 1$. $\Box$

It is clear that $F^\partial[t] \subset \mathrm{Nil}_F(\partial)$. We will use induction on $\deg_\partial(a) = n$ to show the opposite inclusion.

Lemma 2. $\mathrm{Nil}_F(\partial) = F^\partial[t]$.
Proof. If $a \in F$ and $\deg_\partial(a) = 0$ then $a \in F^\partial$ by definition. Let us make a step from $\deg_\partial(a) = n-1$ to $\deg_\partial(a) = n$. If $\deg_\partial(a) = n$ then $\deg_\partial(\partial(a)) = n-1$ and by induction $\partial(a) = \sum_{i=0}^{n-1} a_i t^{n-1-i}$ for some $a_i \in F^\partial$.
Take $f = \sum_{i=0}^{n-1}(n-i)^{-1} a_i t^{n-i}$. Then $\partial(f) = \partial(a)$. So $\partial(a-f) = 0$ which means that $a = f + a_n$ where $a_n \in F^\partial$.

Remark 1. It is clear that $\deg_\partial$ and $\deg_t$ are the same functions. This, of course, gives a proof of the properties of $\deg_\partial$ mentioned above.

Remark 2. A^∂ is algebraically closed in A. Indeed, if $a \notin A^\partial$ then it is represented by a polynomial of positive degree in t and $p(a)$ also has a positive degree in t for any nonzero polynomial p.

Remark 3. Similarly, F^∂ is algebraically closed in F and $F = F^\partial(t)$. Therefore the transcendence degree of F^∂ is the transcendence degree of F minus one: $\mathrm{trdeg}(F^\partial) = \mathrm{trdeg}(F) - 1$. Hence $\mathrm{trdeg}(A^\partial) = \mathrm{trdeg}(A) - 1$.

Lemma 3. F^∂ is the field of fractions of A^∂.
Proof. This proof was suggested by Ofer Hadas. Let $a, b \in A$ and $r = ab^{-1} \in F^\partial$. Assume also that $\deg_\partial(a)$ is minimal possible for all presentations of r as a fraction. Now $\partial(r) = (\partial(a)b - a\partial(b))b^{-2} = 0$. So $ab^{-1} = \partial(a)\partial(b)^{-1}$ and $\deg_\partial(\partial(a)) < \deg_\partial(a)$. To avoid a contradiction we have to assume that $\deg_\partial(a) = 0$, so a and b are in A^∂. $\Box$

Lemma 4. A factorially closed subalgebra A of a ring $\mathbb{C}_n$ of polynomials which has the transcendence degree one is a polynomial ring in one variable.
Proof. Consider a polynomial u of the smallest positive degree in A. Let us assume that $p \in A$ is irreducible. Since A is a subalgebra of transcendence degree one, u and p are algebraically dependent. Let $Q(u,p) = 0$ be an irreducible dependence between them. Then $Q(u,p) = Q(u,0) + pQ_1(u,p)$. Therefore $Q(u,0) = \prod(u - \lambda_i),\ \lambda \in \mathbb{C}$ is divisible by p. Elements $u - \lambda$ are irreducible for any $\lambda \in \mathbb{C}$ because otherwise we will have an element in A with the degree smaller then the degree of u. Since p is irreducible it implies that $p = c(u - \lambda)$ for some $c \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}$. Since each element of A is a product of irreducible elements this subring should be $\mathbb{C}[u]$. $\Box$

Description of lnds of $\mathbb{C}[x, y]$.

Now we will find all locally nilpotent derivations of $\mathbb{C}[x, y]$. If $\partial \in \mathrm{Nil}(\mathbb{C}[x, y]) \setminus 0$ then $\mathbb{C}[x, y]^\partial$ is a subalgebra of transcendence degree one which is a UFD since $\mathbb{C}[x, y]$ is a UFD and $\mathbb{C}[x, y]^\partial$ is factorially closed. Because of that $\mathbb{C}[x, y]^\partial = \mathbb{C}[u]$ (Lemma 4).

Now let us show that $D(f) = \mathrm{J}(u, f)$ is an lnd which is equivalent to ∂. Extend ∂ on $\mathbb{C}(x, y)$. Then $\mathrm{Nil}_{C(x,y)}(\partial) = \mathbb{C}(u)$ and $\mathbb{C}[x, y] \subset \mathbb{C}(u)[t]$ where $\partial(t) = 1$ and $t = \frac{r}{q(u)}$, $r \in \mathbb{C}[x, y]$. Therefore $x = \sum x_i t^i$, $y = \sum y_j t^j$ where x_i, $y_j \in \mathbb{C}(u)$ and $1 = \mathrm{J}(x, y) = \mathrm{J}(u, t)\mathrm{J}_{u,t}(\sum x_i t^i, \sum y_j t^j)$. Now, $\mathrm{J}(u, t) = \mathrm{J}(u, \frac{r}{q(u)}) = \frac{\mathrm{J}(u,r)}{q(u)} \in \mathbb{C}(u)[t]$ and $\mathrm{J}_{u,t}(\sum x_i t^i, \sum y_j t^j) \in \mathbb{C}(u)[t]$. Hence these two polynomials in t have degree zero and $\mathrm{J}(u, r) \in \mathbb{C}[u]$, $\mathrm{J}(u, t) \in \mathbb{C}(u)$. Thus D is an lnd which is equivalent ∂.

After that, description of u is as in the proof of Jung theorem. Indeed, we checked that if $D(f) = \mathrm{J}(u, f)$ is an lnd then the Newton polygon of u is a right triangle with the right angle in the origin and that the leading form which corresponds to the hypotenuse is either $\mu(y - cx^{m_1})^k$ or $\mu(y^{n_1} - c_i x)^k$ where $\mu \in \mathbb{C}$. In any case, we can make an automorphism which will make this triangle smaller. As before we can conclude that there is an automorphism γ such that the Newton polygon of $\gamma(u)$ is not a triangle anymore and belongs to one of the coordinate axes. If $\gamma(u) = p(x)$ then $u = p(\gamma^{-1}(x))$ and since $\mathbb{C}[x, y]^\partial = \mathbb{C}[u]$ we can assume that $u = \gamma^{-1}(x)$. Similarly, if $\gamma(u) = p(y)$ then $u = p(\gamma^{-1}(y))$ and we can assume that $u = \gamma^{-1}(y)$. Therefore u is a generator of $\mathbb{C}[x, y]$, i.e. there exists $v \in \mathbb{C}[x, y]$ such that $\mathbb{C}[u, v] = \mathbb{C}[x, y]$. Since we can assume that $\mathrm{J}(u, v) = 1$ the derivation D is just the partial derivative relative to v. Since ∂ is equivalent to D we have $\partial(v) = p(u)$ and $\partial = p(u)\frac{\partial}{\partial v}$.

Finally, if ∂ is an lnd of $\mathbb{C}[x, y]$ then $\alpha_T = \exp(T\partial) = \sum_{i=0}^{\infty} \frac{(T\partial)^i}{i!}$ where T is "time" defines a shift of the plane $\mathbb{C}^2$ in a direction of a line which is isomorphic to a straight line: a point (x, y) moves to the point $(x, y)_T = (\alpha_T(x), \alpha_T(y))$.

For example, if $\partial(f) = \mathrm{J}(x + y^2, f)$ then $(x, y)^T = (x - 2Ty - T^2, y + T)$ and if we consider all trajectories of point the plane will be covered by lines isomorphic to, say, the x axis.

We will see in the third lecture that if a line which is isomorphic to the x axis is embedded into a plane then it can be included as one of the trajectories in a similar picture.